THE FORCING COMPANIONS OF NUMBER THEORIES

BY

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ABSTRACT

This paper is concerned with the finite forcing companion T^f and the infinite forcing companion T^F of a number theory T. A number theory is any theory containing the $\forall_2$-part of peano number theory P. Two of our results are as follows: (A) for each number theory T, the theory T^f is not arithmetical, and the theory T^F is not analytical, and (B) there is a sentence $\sigma \in \forall_4$ such that, for each two (not necessarily distinct) number theories T_1, T_2, both $\sigma \in T_1^f$ and $\neg\, \sigma \in T_2^F$.

0. Introduction

Although our results are concerned with forcing companions, we do not use any forcing techniques in our proofs. In fact our proofs can be followed (although not necessarily understood) without any knowledge of model-theoretic forcing. This is because the relevant facts concerning model-theoretic forcing can be described in a standard model-theoretic setting. This is done in §1.

Our proofs make use of a construction of M. Rabin concerning P. We show that in certain nonstandard models of number theories, the set of standard integers can be defined. This is done in §2.

In §3 we combine §1 and §2 to get our results, and in §4 we give some open questions.

The results given here were obtained during October–December 1971 while the first author was spending some time at the University of Aberdeen. This was after the second author had heard A. Robinson speak on the subject at the International Congress for Logic, Methodology, History and Philosophy of Science,

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in Bucharest, September 1971, but before any of us had seen Robinson's paper [13]. Our results considerably improve Robinson's. (After receiving a preprint of [13] we were able to simplify part of §2; however, this made no essential improvements to our results.) ·

In February 1972, we received summaries of the doctoral dissertations of J. Hirschfeld [3], and W. H. Wheeler [18]. Hirschfeld's dissertation contains some results similar to ours. After reading the summary of [3], our understanding of model-theoretic forcing as applied to number theories has greatly improved, however the results given here are not influenced by [3].

We recommend [3] as essential reading for everyone interested in the subject.

Wheeler has obtained several results concerning the complexities of forcing companions, and analogues of some of our results for theories of groups and division rings (see also [5], [6]).

1. The model-theoretic requirements

In this section we set up the required model-theoretic machinery. Nowhere in the section do we use or again mention number theories.

We look at some fixed but arbitrary countable first order language L; all theories, structures, etc., that occur are L-theories, L-structures, etc. (The countability of L is necessary for our discussion of finite forcing, and simplifies slightly our discussion of infinite forcing.) The sets of L-formulae $\forall_1$, $\exists_1$, $\forall_2$, $\exists_2, \cdots$ are defined in the usual way as sets of formulae in prenex normal form with specified prenexes, as indicated.

A model of a theory T is, of course, a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \mathrm{T}$; let $\mathscr{M}(\mathrm{T})$ be the class of models of T. A submodel of T is a structure $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \in \mathscr{M}(\mathrm{T})$; let $\mathscr{S}(\mathrm{T})$ be the class of submodels of T. Thus $\mathscr{S}(\mathrm{T})$ is the class of models of $\mathrm{T}_\cap \forall_1$.

Two theories T_1, T_2 are mutually model-consistent if $\mathscr{S}(\mathrm{T}_1) = \mathscr{S}(\mathrm{T}_2)$, in other words if $\mathrm{T}_{1\cap} \forall_1 = \mathrm{T}_{2\cap} \forall_1$.

A class of structures $\mathscr{K}_1$ is cofinal in a class of structures $\mathscr{K}_2$ if $\mathscr{K}_1 \subseteq \mathscr{K}_2$ and, for each $\mathfrak{A} \in \mathscr{K}_2$, there is some $\mathfrak{B} \in \mathscr{K}_1$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Thus $\mathscr{M}(\mathrm{T})$ is cofinal in $\mathscr{S}(\mathrm{T})$.

The basic concept of this section is that of a companion operator.

Definition. A companion operator is a function $(-)^*$ taking theories T to theories T^* such that

(*1) $T_{1\cap}\forall_1 = T_{2\cap}\forall_1 \Rightarrow T_1^* = T_2^*$,

(*2) $T_\cap \forall_1 = T^*{}_\cap \forall_1$,

(*3) $T_\cap \forall_2 \subseteq T^*$.

The theory T* is the *-companion of T, and T is *-complete if T = T*.

A standard method of producing a companion operator is to select a cofinal subclass $\mathscr{K}$ of each $\mathscr{S}$(T) and put T* = Th($\mathscr{K}$). The conditions (*1) and (*2) are then automatically satisfied, and by a suitable choice of $\mathscr{K}$ we have (*3) also satisfied.

Historically, the notion of a companion operator goes back to that of the model completion of a theory, as defined by A. Robinson. This was later refined by E. Bers to the notion of the model companion of a theory; a theory T^m is a model companion of a theory T if T, T^m are mutually model-consistent and T^m is model-complete. An easy proof shows that each theory has at most one model companion, and clearly any model completion is a model companion. However there are theories with no model companion, and theories with a model companion but no model completion. (It is known that a theory T has a completion if and only if it has a companion and $T_\cap \forall_1$ has the amalgamation property, in which case the companion is the completion. Since a theory T has quantifier elimination if an only if T is model-complete and $T_\cap \forall_1$ has the amalgation property, we see that the existence of a completion gives us quantifier elimination but the existence of a mere companion does not.)

If T has a model companion then the action on T of any companion operator (–)* is trivial (see Theorem 1.1 below). If T has no model companion then T* is a possible substitute for T^m.

THEOREM 1.1. *If the theory* T *has a model companion* T^m *then* $T^* = T^m$ *for each companion operator* (–)*.

PROOF. T^m is $\forall_2$-axiomatizable and the theories T^m, T are mutually model-consistent; thus (*3) and (*1) give

$$T^m \subseteq T^{m*} = T^*.$$

Now consider any $\mathfrak{A} \models T^m$. We have $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \models T^*$ (because of (*2)), and so $\mathfrak{A} \prec \mathfrak{B}$ (since T^m is model-complete). Thus $\mathfrak{A} \models T^*$ and so $T^* \subseteq T^m$, as required.

An attempt was made by K. Kaiser in [4] to deal with theories without model companions. Essentially, he introduced what turns out to be the minimum companion operator, by proving the following theorem.

THEOREM 1.2. *For each theory* T, *the set* $K(\mathrm{T})$, *of* $\forall_2$*-axiomatizable theories* T′ *such that* T, T′ *are mutually model-consistent, has a maximum member* T^0.

PROOF. (See [4, Sätze 1, 2, 3].) Zorn's lemma produces a maximal member T^0 of $K(\mathrm{T})$, and for any $\mathrm{T}' \in K(\mathrm{T})$, the deductive closure of $\mathrm{T}^0 \cup \mathrm{T}'$ is also in $K(\mathrm{T})$; hence, $\mathrm{T}' \subseteq \mathrm{T}^0$ and T^0 is maximum.

COROLLARY 1.3. *The function* $(-)^0$ *is a companion operator.*

PROOF. (*1) Suppose T_1, T_2 are mutually model-consistent. Let $\mathrm{T}' = \mathrm{T}_2^0$, so that T′ is $\forall_2$-axiomatizable and (using T_2) the theories T_1, T′ are mutually model-consistent. Thus $\mathrm{T}_2^0 = \mathrm{T}' \subseteq \mathrm{T}_1^0$, and similarly $\mathrm{T}_1^0 \subseteq \mathrm{T}_2^0$.

(*2) This follows immediately from the definition of T^0.

(*3) This follows by letting T′ be the deductive closure of $\mathrm{T} \cap \forall_2$.

To see that $(-)^0$ is the minimum companion operator, we use the following theorem.

THEOREM 1.4. *For each two companion operators* $(-)^*$, $(-)^{*\prime}$ *and theory* T,

i) $\mathrm{T}^{**\prime} = \mathrm{T}^{*\prime}$,

ii) $\mathrm{T}^* \cap \forall_2 = \mathrm{T}^{*\prime} \cap \forall_2$,

iii) $\mathrm{T}^0 \subseteq \mathrm{T}^*$.

PROOF. Part (i) follows since T, T* are mutually model consistent; (ii) follows since

$$\mathrm{T}^* \cap \forall_2 \subseteq \mathrm{T}^{**\prime} = \mathrm{T}^{*\prime};$$

and (iii) follows from (ii) since T^0 is $\forall_2$-axiomatizable.

It is possible to describe the $\exists_1$-part of T^0; to do this let us say a sentence σ is T-tame if $\sigma \in \exists_1$ and

$$\mathrm{T} \vdash \sigma \rightarrow \alpha \Rightarrow \mathrm{T} \vdash \alpha$$

holds for each sentence $\alpha \in \forall_1$.

THEOREM 1.5. *For each theory* T, $\mathrm{T}^0 \cap \exists_1$ *is the set of* T*-tame sentences.*

PROOF. Consider any sentence $\sigma \in \exists_1$.

Suppose first that $\sigma \in \mathrm{T}^0$ and $\mathrm{T} \vdash \sigma \rightarrow \alpha$ for some sentence $\alpha \in \forall_1$. The sentence $\sigma \rightarrow \alpha$ is $\forall_2$ and so $\mathrm{T}^0 \vdash \sigma \rightarrow \alpha$ which gives $\mathrm{T}^0 \vdash \alpha$. Thus $\mathrm{T} \vdash \alpha$, and so σ is T-tame.

Secondly, suppose that σ is T-tame. Notice that $T \cup \{\sigma\}$ is consistent (for, if not, then

$$T \vdash \sigma \to (\forall v)[v \neq v]$$

and so T is inconsistent). Let T_1 be the deductive closure of $T \cup \{\sigma\}$. The tameness of σ shows that T, T_1 are mutually model-consistent, and so $\sigma \in T_1^0 = T^0$, as required.

We now turn our attention to the two forcing companion operators $(-)^f$, $(-)^g$ introduced by A. Robinson in a series of papers [1, 10, 11, 12]. (We write $(-)^g$ for Robinson's $(-)^F$ in order to simplify the notation.) Originally, $(-)^f$, $(-)^g$ were constructed using forcing techniques; however, it turns out that they can be described in standard model-theoretic terms. To do this, it is convenient to consider another companion operator $(-)^e$.

The relevant facts concerning T^e, T^g are given in Theorems 1.6–1.11 (below). These theorems are not new here; they have been known to several people (including G. Cherlin, E. Fisher, and A. Robinson) for some time. There are no adequate references for their proofs, but, with the aid of the given hints, the reader should have no difficulty providing their proofs.

The relevant facts concerning T^f are given in Theorem 1.12 and its Corollaries 1.13–1.15.

Let $\mathfrak{A}$ be any structure. By a type over $\mathfrak{A}$ we will mean here a set $\Gamma(\bar{a}, \bar{v})$ of $\exists_1$-formulae containing parameters $\bar{a}$ (denoting members of $\mathfrak{A}$) and free variables $\bar{v}$. We are particularly interested in finite and countable types. For any two structures $\mathfrak{A}$, $\mathfrak{B}$ we write

$$\mathfrak{A} \prec_1 \mathfrak{B} \quad \text{or} \quad \mathfrak{A} \prec\prec_1 \mathfrak{B}$$

if $\mathfrak{A} \subseteq \mathfrak{B}$ and each finite or countable (respectively) type over $\mathfrak{A}$ which is realized in $\mathfrak{B}$ is already realized in $\mathfrak{A}$.

Definition. For each theory T, the class $\mathscr{E}_T$ is the class of submodels $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

holds for each model $\mathfrak{B}$ of T.

The class $\mathscr{C}_T$ is the class of submodels $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec\prec_1 \mathfrak{B}$$

holds for each model $\mathfrak{B}$ of T.

The class $\mathscr{E}_{\mathrm{T}}$ has been studied in a slightly different context in [15]; the members of $\mathscr{E}_T$ are T-existentially closed structures. The class $\mathscr{C}_{\mathrm{T}}$ has been studied in [17], where the members of $\mathscr{C}_{\mathrm{T}}$ are called $\aleph_1 - WJ_0$ structures.

We have the following characterizations of $\mathscr{E}_{\mathrm{T}}, \mathscr{C}_{\mathrm{T}}$.

THEOREM 1.6. *For each theory* T, *the class* $\mathscr{E}_{\mathrm{T}}$ *is uniquely characterized by the following three conditions*:

i) $\mathscr{E}_{\mathrm{T}}$ *is cofinal in* $\mathscr{S}(\mathrm{T})$.

ii) *The implication*

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

holds for each two members $\mathfrak{A}, \mathfrak{B}$ *of* $\mathscr{E}_{\mathrm{T}}$.

iii) *The implication*

$$\mathfrak{A} \prec_1 \mathfrak{B} \in \mathscr{E}_{\mathrm{T}} \Rightarrow \mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$$

holds for each two structures $\mathfrak{A}, \mathfrak{B}$.

PROOF. See [15, Th. 2.8, 2.10].

THEOREM 1.7. *For each theory* T *the class* $\mathscr{C}_{\mathrm{T}}$ *is uniquely characterized by the following three conditions*:

i) $\mathscr{C}_T$ *is cofinal in* $\mathscr{S}(\mathrm{T})$.

ii) *The implication*

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec\prec_1 \mathfrak{B}$$

holds for reach two membes $\mathfrak{A}, \mathfrak{B}$ *of* $\mathscr{C}_{\mathrm{T}}$.

iii) *The implication*

$$\mathfrak{A} \prec\prec_1 \mathfrak{B} \in \mathscr{C}_{\mathrm{T}} \Rightarrow \mathfrak{A} \in \mathscr{C}_{\mathrm{T}}$$

holds for each two structures $\mathfrak{A}, \mathfrak{B}$.

PROOF. For existence, see [17, Th. 2.8] and then argue as in Theorem 1.6.

The connection between $\mathscr{E}_{\mathrm{T}}, \mathscr{C}_{\mathrm{T}}$ is given in the following theorem.

THEOREM 1.8. *For each theory* T, $\mathscr{E}_{\mathrm{T}}$ *is the class of structures* $\mathfrak{A}$ *such that* $\mathfrak{A} \prec_1 \mathfrak{B}$ *for some* $\mathfrak{B} \in \mathscr{C}_{\mathrm{T}}$.

We also note the following theorem.

THEOREM 1.9. *For each theory* T *the implication*

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

holds for each two members $\mathfrak{A}, \mathfrak{B}$ *of* $\mathscr{C}_T$.

PROOF. See [17, Corollary 5.6].

For each theory T, we let

$$T^e = \mathrm{Th}(\mathscr{E}_T), \ T^g = \mathrm{Th}(\mathscr{C}_T)$$

and obtain two companion operators.

THEOREM 1.10. *The two functions* $(-)^e$, $(-)^g$ *are companion operators.*

PROOF. From Theorem 1.6, we see that $(-)^e$ has (*1) and (*2). For each $\mathfrak{A} \in \mathscr{E}_T$, we have $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \models T$ (since $\mathscr{E}_T$ is cofinal in $\mathscr{S}(T)$). But then $\mathfrak{A} \prec_1 \mathfrak{B}$ and so $\mathfrak{A} \models T_\cap \forall_2$, hence $(-)^e$ has (*3).

In the same way we see that $(-)^g$ is a companion operator.

The theory T^g is the infinite forcing companion of T; to see this we go via the class of T-infinite generic structures.

DEFINITION. For each theory T, the class $\mathscr{G}_T$ is the class of submodels $\mathfrak{A}$ of T such that $\mathfrak{A} \prec \mathfrak{B}$ for some $\mathfrak{B} \in \mathscr{C}_T$.

The following theorem characterizes $\mathscr{G}_T$.

THEOREM 1.11. *For each theory* T, *the class* $\mathscr{G}_T$ is *uniquely characterized by the following three conditions*:

i) $\mathscr{G}_T$ *is cofinal in* $\mathscr{S}(T)$.

ii) *The implication*

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

holds for each two members $\mathfrak{A}, \mathfrak{B}$ *of* $\mathscr{G}_T$.

iii) *The implication*

$$\mathfrak{A} \prec \mathfrak{B} \in \mathscr{G}_T \Rightarrow \mathfrak{A} \in \mathscr{G}_T$$

holds for each two structures $\mathfrak{A}, \mathfrak{B}$.

The class of T-infinite-generics satisfies conditions (i), (ii) and (iii) of Theorem 1.11 and so $\mathscr{G}_T$ is this class. Clearly, $T^g = \mathrm{Th}(\mathscr{G}_T)$ so that T^g is the infinite forcing companion of T (i.e. Robinson's T^F).

Several people have noted that the whole of the machinery of infinite forcing, can be developed from this standpoint.

We now come to the finite forcing companion operator.

DEFINITION. For each theory T, the class $\mathscr{F}_T$ is the class of T-finite-generic structures, and $T^f = \mathrm{Th}(\mathscr{F}_T)$.

The development of the properties of $\mathscr{F}_T$ and T^f can be found in [1]. For us the following characterization of $\mathscr{F}_T$ (Theorem 1.12) is sufficient. Remember that a completing model of a theory T is a model $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

holds for all models $\mathfrak{B}$ of T .

THEOREM 1.12. *For each theory* T *the class* $\mathscr{F}_T$ *is uniquely characterized by the following two conditions*:

i) T, $\mathrm{Th}(\mathscr{F}_T)$ *are mutually model-consistent.*

ii) $\mathscr{F}_T$ *is the class of completing models of* $\mathrm{Th}(\mathscr{F}_T)$.

PROOF. See [16].

COROLLARY 1.13. *The function* $(-)^f$ *is a companion operator.*

COROLLARY 1.14. *A theory* T *is f-complete if and only if* T *is the theory of the class of its completing models.*

COROLLARY 1.15. *For each theory* T, *if* $T^f \subseteq T$ then $T^f = T$.

Notice that for each theory T we have

$$\mathscr{C}_T \subseteq \mathscr{G}_T \subseteq \mathscr{E}_T \supseteq \mathscr{F}_T,$$
$$T^g \supseteq T^e \subseteq T^f.$$

Let B be some fixed $\forall_2$-axiomatizable theory and put

$$\mathbf{B} = \{T : B \subseteq T\}$$

i.e. **B** is the set of theories extending B. This set **B** is closed under each companion operator $(-)^*$; in particular, **B** is closed under $(-)^e$, $(-)^f$, $(-)^g$. Associated with B are three theories E, F, G and four classes of models $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{C}$ given as follows:

$$\begin{aligned} E &= \cap\{T^e : T \in \mathbf{B}\}, & \mathscr{E} &= \cup\{\mathscr{E}_T : T \in \mathbf{B}\},\\ F &= \cap\{T^f : T \in \mathbf{B}\}, & \mathscr{F} &= \cup\{\mathscr{F}_T : T \in \mathbf{B}\},\\ G &= \cap\{T^g : T \in \mathbf{B}\}, & \mathscr{G} &= \cup\{\mathscr{G}_T : T \in \mathbf{B}\},\\ & & \mathscr{C} &= \cup\{\mathscr{C}_T : T \in \mathbf{B}\}. \end{aligned}$$

Clearly we have,

$$\mathscr{C} \subseteq \mathscr{G} \subseteq \mathscr{E} \supseteq \mathscr{F},$$

$$\mathrm{E} = \mathrm{Th}(\mathscr{E}),\ \mathrm{F} = \mathrm{Th}(\mathscr{F}),$$

$$\mathrm{G} = \mathrm{Th}(\mathscr{G}) = \mathrm{Th}(\mathscr{C}),$$

and for $\mathrm{T} \in \mathbf{B}$,

$$\begin{array}{ccc} \mathrm{F} & \subseteq & \mathrm{T}^f \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{E} & \subseteq & \mathrm{T}^e \\ \mid\!\cap & & \mid\!\cap \\ \mathrm{G} & \subseteq & \mathrm{T}^g. \end{array}$$

The classes $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{C}$ are characterized in the following theorem.

THEOREM 1.16. 1) *The members of $\mathscr{E}$ are exactly those models $\mathfrak{A}$ of* B *such that*

$$\mathfrak{A} \subseteq \mathfrak{B},\ \mathfrak{A} \equiv \mathfrak{B} \Rightarrow \mathfrak{A} \prec_1 \mathfrak{B}$$

holds for all structures $\mathfrak{B}$.

2) *The members of $\mathscr{F}$ are exactly those models $\mathfrak{A}$ of* B *such that*

$$\mathfrak{A} \subseteq \mathfrak{B},\ \mathfrak{A} \equiv \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

holds for all structures $\mathfrak{B}$.

3) *The members of $\mathscr{G}$ are exactly those models $\mathfrak{A}$ of* B *such that $\mathfrak{A} \prec \mathfrak{B}$ for some $\mathfrak{B} \in \mathscr{C}$.*

4) *The members of $\mathscr{C}$ are exactly those models $\mathfrak{A}$ of $\mathfrak{B}$ such that*

$$\mathfrak{A} \subseteq \mathfrak{B},\ \mathfrak{A} \equiv \mathfrak{B} \Rightarrow \mathfrak{A} \prec\prec_1 \mathfrak{B}$$

holds for all structures $\mathfrak{B}$.

PROOF. 1) First, suppose that $\mathfrak{A} \in \mathscr{E}$, so that $\mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$ for some $\mathrm{T} \in \mathbf{B}$. Consider any $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$. We have $\mathfrak{B} \in \mathscr{S}(\mathrm{T})$ so that $\mathfrak{B} \subseteq \mathfrak{C}$ for some $\mathfrak{C} \models \mathrm{T}$. Thus $\mathfrak{A} \prec_1 \mathfrak{C}$ which gives $\mathfrak{A} \prec_1 \mathfrak{B}$.

Secondly, suppose $\mathfrak{A}$ satisfied the implication of (1) for all structures $\mathfrak{B}$. We have $\mathrm{T} = \mathrm{Th}(\mathfrak{A}) \in \mathbf{B}$ and $\mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$, hence $\mathfrak{A} \in \mathscr{E}$, as required.

2) First, suppose that $\mathfrak{A} \in \mathscr{F}$. We see that $\mathfrak{A}$ satisfies the required implication by the method of part (1) (using Theorem 1.12).

Secondly, suppose that $\mathfrak{A}$ satisfies the given implication and let $T = Th(\mathfrak{A})$. Corollary 1.14 shows that T is f-complete, and so $\mathfrak{A} \in \mathscr{F}_T$, as required.

3) This follows from the definition of $\mathscr{G}_T$.

4) This is proved in the same way as (1).

Finally we note the following theorem.

THEOREM 1.17. *The theories* B, E, F, G *have the same* $\forall_1$ *parts.*

PROOF. Consider any sentence $\alpha \in \forall_1$. If $\alpha \notin B$, then $B \cup \{\neg \alpha\}$ is consistent, and so

$$\neg \alpha \in T^e \subseteq T^f \cap T^g$$

where T is the deductive closure of $B \cup \{\neg \alpha\}$. Thus

$$E \cup \{\neg \alpha\}, \; F \cup \{\neg \alpha\}, \; G \cup \{\neg \alpha\}$$

are each consistant, so that

$$\alpha \notin E, \; \alpha \notin F, \; \alpha \notin G.$$

This gives

$$E \cap \forall_1 \subseteq B \cap \forall_1, \; F \cap \forall_1 \subseteq B \cap \forall_1, \; G \cap \forall_1 \subseteq B \cap \forall_1,$$

and since the reverse inclusions are trivial, we have the required result.

COROLLARY 1.18. *For each theory* $T \in \mathbf{B}$, *the following implications hold:*

i) $T^e \subseteq E^e \Rightarrow T^e = E^e$.

ii) $T^f \subseteq F^f \Rightarrow T^f = F^f$.

iii) $T^g \subseteq G^g \Rightarrow T^g = G^g$.

PROOF. In each case, the hypothesis shows that T, **B** are mutually model-consistent.

2. The number-theoretic requirements

Let L be any countable language in which number theory can be formalized. The particular form of L is not important; however, we will make explicit use of the L-numerals $0, 1, 2, \cdots, n, \cdots$, and the L-formula "$v < w$" (which is assumed to be $\exists_1$). We let $\mathfrak{N}$ be the L-structure with carrier set ω (the set of natural numbers) such that $Th(\mathfrak{N})$ is full (complete) number theory.

Let P be peano number theory, suitably formalized in L.

We study certain L-theories and L-structures called number theories and number structures, respectively.

DEFINITION. The basic number theory B is the theory axiomatized by $\mathbf{P} \cap \forall_2$. A number theory is any theory extending B (i.e. any member of **B**). A number structure is any model of **B**.

Notice that each number structure contains an isomorphic copy of $\mathfrak{N}$ as an initial segment. We will conveniently confuse $\mathfrak{N}$ with this initial segment.

Associated with B, we have all the machinery of §1; in particular, we have the class $\mathscr{E}$ of number structures. The aim of this section is to prove the following theorem.

THEOREM 2.1. *There is a certain $\exists_3$-formula $I(x)$ (containing just one free variable x) such that for each $\mathfrak{A} \in \mathscr{E}$,*

i) $I^{\mathfrak{A}} = \omega$,

ii) $\mathfrak{A} = \mathfrak{N} \Leftrightarrow \mathfrak{A} \models \rho$,

where ρ is the $\forall_4$-sentence $(\forall x)I(x)$.

In part (i) of this theorem

$$I^{\mathfrak{A}} = \{a \in A : \mathfrak{A} \models I(a)\}$$

where A is the carrier set of $\mathfrak{A}$.

The theory B contains quite a lot of recursive function theory. For us, the two important facts are concerned with the enumeration of r.e. and finite subsets of ω.

Let $W_0, W_1, W_2, \cdots$ be any r.e. enumeration of all r.e. sets. Then, making essential use of Matijasevič's theorem [8], we have some $\exists_1$-formula $d(v, w)$ such that for each $m \in \omega$,

$$n \in W_m \Leftrightarrow \mathrm{B} \models d(n, m)$$

holds for all $n \in \omega$. The formula $d(v, w)$ can be chosen so that for each formula $\theta(v) \in \exists_1$,

$$\mathrm{B} \vdash (\forall v)[\theta(v) \leftrightarrow d(v, t)]$$

holds for some $t \in \omega$.

This formula $d(v, w)$ will remain fixed throughout.

Let $F_0, F_1, F_2, \cdots$ be any standard enumeration of all finite sets. Then there are formulae $f_+(v, w) \in \exists_1$, $f_-(v, w) \in \forall_1$ such that for each $m \in \omega$,

$$n \in F_m \Leftrightarrow \mathrm{B} \vdash f_+(n, m),$$

$$n \notin F_m \Leftrightarrow \mathrm{B} \vdash \neg f_-(n, m),$$

hold for all $n \in \omega$. These two formulae can be chosen so that

$$B \vdash (\forall v, w)[f_+(v,w) \leftrightarrow f_-(v,w)].$$

Throughout, we let $f(v,w)$ be either $f_+(v,w)$ or $f_-(v,w)$; at most times, it will not matter which one $f(v,w)$ is.

The following lemma is well known (c.f. [9, Th. 2.1]).

LEMMA 2.2. i)

$$P \vdash (\forall w_1, w_2)(\exists w)(\forall v)[d(v,w_1) \vee d(v,w_2) \leftrightarrow d(v,w)].$$

ii) *For each $m_1, m_2 \in \omega$, there is some $m \in \omega$ such that*

$$B \vdash (\forall v)[d(v,m_1) \vee d(v,m_2) \leftrightarrow d(v,m)].$$

To get the formula $I(x)$, we combine a construction of Rabin [9], with one of Robinson [13]. Let $\delta(v)$ be any $\forall_1$-formula such that

$$P \vdash (\forall w) \neg (\forall v)[\delta(v) \leftrightarrow d(v,w)] \tag{2.1}$$

and for each number structure $\mathfrak{A}$,

$$\mathfrak{A} \models \neg(\forall v)[\delta(v) \leftrightarrow d(v,m)] \tag{2.2}$$

holds for all $m \in \omega$. (For instance we can put $\delta(v) = \neg d(v,v)$.) Now consider the formulae

$$H(w) = (\forall v)[d(v,w) \to \delta(v)],$$

$$K(v,x) = (\exists w < x)[H(w) \vee d(v,w)],$$

so that $K(v,x) \in \exists_2$. Making use of Lemma 2.2, we have the following lemma (c.f. [9, Th. 2.2]).

LEMMA 2.3. i) $P \vdash (\forall x)(\exists w)(\forall v)[d(v,w) \leftrightarrow K(v,x)]$.

ii) *For each number structure $\mathfrak{A}$ and each $n \in \omega$, there is some $m \in \omega$ such that*

$$\mathfrak{A} \models (\forall v)[d(v,m) \leftrightarrow K(v,n)].$$

At this point we could follow Rabin and put

$$I(x) = (\exists w)(\forall v)[d(v,w) \leftrightarrow K(v,x)].$$

This gives Theorem 2.1 except that this $I(x)$ is $\exists_4$ rather than $\exists_3$. In order to get $I(x) \in \exists_3$, we put

$$J(x) = (\forall v)[\delta(v) \leftrightarrow K(v,x)]$$

(c.f. [13, 2.7]) and let $I(x) = \neg J(x)$. This $I(x)$ is $\exists_3$.

THEOREM 2.4. i) $\mathrm{P} \vdash (\forall x)I(x)$.

ii) *For each number structure* $\mathfrak{A}$ *and each* $n \in \omega$, $\mathfrak{A} \models I(n)$.

PROOF. i) Consider any model $\mathfrak{A}$ of P and some $a \in A$. Let a' be the successor of a in $\mathfrak{A}$.

First, suppose that $\mathfrak{A} \models H(a)$, so that

$$\mathfrak{A} \models (\forall v)[K(v,a') \leftrightarrow K(v,a) \vee d(v,a)],$$

and hence Lemma 2.3(i) gives

$$\mathfrak{A} \models (\forall v)[K(v,a') \leftrightarrow d(v,b) \vee d(v,a)]$$

for some $b \in A$. Lemma 2.2(i) now gives

$$\mathfrak{A} \models (\forall v)[K(v,a') \leftrightarrow d(v,c)]$$

for some $c \in A$.

This is still true if $\mathfrak{A} \models \neg H(a)$, for in this case we can put $c = b$.

Thus we have

$$\mathfrak{A} \models J(a') \leftrightarrow (\forall v)[\delta(v) \leftrightarrow d(v,c)]$$

so that (2.1) gives $\mathfrak{A} \models I(a')$. It is trivial to verify that $\mathfrak{A} \models I(0)$, and so $\mathfrak{A} \models (\forall x)I(x)$, as required.

ii) This is proved in the same way using Lemmas 2.3(ii), 2.2(ii) and (2.2).

We are now ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1. i) Consider any $\mathfrak{A} \in \mathscr{E}$. Theorem 2.4(ii) gives $\omega \subseteq I^{\mathfrak{A}}$, hence it suffices to show $I^{\mathfrak{A}} \subseteq \omega$. To do this we show $A - \omega \subseteq J^{\mathfrak{A}}$.

Consider any $a \in A - \omega$. Since

$$\mathfrak{A} \models (\forall v)[K(v,a) \rightarrow \delta(v)]$$

holds for all number structures it is sufficient to show

$$\mathfrak{A} \models (\forall v)[\delta(v) \rightarrow K(v,a)]. \tag{2.3}$$

Suppose $\mathfrak{A} \models \delta(b)$ for some $b \in A$. Since $\delta(v) \in \forall_1$ and $\mathfrak{A} \in \mathscr{E}$, a simple compactness argument (c.f. [15, Th. 2.1]) produces a formula $\theta(v) \in \exists_1$ such that

$$\mathfrak{A} \models \theta(b), \quad \mathfrak{A} \models (\forall v)[\theta(v) \rightarrow \delta(v)].$$

We may replace $\theta(v)$ by $d(v,t)$ for some $t \in \omega$ so that

$$\mathfrak{A} \models d(b,t), \quad \mathfrak{A} \models H(t).$$

Thus, since $a \in A - \omega$ and so $t < a$, we have $\mathfrak{A} \models K(b, a)$, which verifies (2.3).

ii) This follows immediately from (i).

Originally, Rabin used his construction to show the independence of the induction axioms from the other axioms of number theory. The methods of this section give the following improvement of [9, theorem on p. 299] and [2, theorem on p. 43].

THEOREM 2.5. *There is a sentence $\rho_n \in \forall_{n+4}$ such that for each $\forall_{n+2}$-axiomatizable number theory* T, $\rho_n \in \mathrm{P} - \mathrm{T}$.

PROOF. First consider $n = 0$. Let $\rho_0 = \rho$ so that $\rho \in \mathrm{P}$ (by Theorem 2.4(i)). Let $\mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$ with $\mathfrak{A} \neq \mathfrak{N}$, so that $\mathfrak{A} \models \neg \rho_0$ (by Theorem 2.1(ii)). But T is $\forall_2$-axiomatizable and so $\mathfrak{A} \models \mathrm{T}$, hence $\rho_0 \notin \mathrm{T}$, as required.

For $n > 0$, the relevant parts of this section and Section 1 can be relativized. Thus $d(v, w)$ is replaced by a formula $d^{(n)}_{(v, w)} \in \exists_{n+1}$ which enumerates all $\exists_{n+1}$-formulae, and $\mathscr{E}$ is replaced by the class of number structures $\mathfrak{A}$ satisfying

$$\mathfrak{A} \prec_n \mathfrak{B}, \ \mathfrak{A} \equiv \mathfrak{B} \rightarrow \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

for all structures $\mathfrak{B}$. (The relations "$\prec_n$", "$\prec_{n+1}$" are defined in the obvious way.) We do *not* need to replace B by $\mathrm{P} \cap \forall_{n+2}$ since $d^{(n)}(v, w)$ can be constructed from $d(v, w)$ using machinery available in **B**.

3. The results

We now use Theorem 2.1 to obtain information about the set up described in §1 with B as the basic number theory. Two particular members of **B** are peano number theory P and full number theory N ($=\mathrm{Th}(\mathfrak{N})$). We assume the language of B has been equipped with a Gödel numbering, so we may describe the complexity of theories by recursion theoretic methods. All of the recursion theory we use can be found in [14].

The following theorem may be well known; however, it does not seem to have been stated anywhere.

THEOREM 3.1. *There is no model-complete number theory.*

PROOF. Suppose T is a model complete number theory, so that $\mathscr{E}_{\mathrm{T}} = \mathscr{M}(\mathrm{T})$ and consider the set of formulae

$$\{I(x)\} \cup \{n < x : n \in \omega\}.$$

This set is finitely satisfiable in every model of T, and so is satisfiable in some model $\mathfrak{A}$ of T. We then have $I^{\mathfrak{A}} \neq \omega$ (since the set is satisfied), and $I^{\mathfrak{A}} = \omega$ (since $\mathfrak{A} \in \mathscr{E}$). This contradiction gives the required result.

The sentence CON(B) which naturally expresses the consistency of B is $\forall_1$ (by [8]), and so is a member of N^0 (since $\mathfrak{N} \models$ CON(B)). In contrast we have the following theorem.

THEOREM 3.2. *The sentence* $\neg$ CON(B) *is* B-*tame, and hence is a member of* B^0.

PROOF. See [7, Th. 4.3] and Theorem 1.5.

This theorem exemplifies the strange properties of companions of number theories. The theorem remains true if B is replaced by any r.e. number theory.

THEOREM 3.3. *For each number theory* T, *both* $T^0 \nsubseteq F$ *and* $T^0 \nsubseteq G$ *hold.*

PROOF. The proof of both parts is similar, so we prove only $T^0 \nsubseteq G$.

Suppose that $T^0 \subseteq G$, so that Theorem 1.17 gives

$$T \cap \forall_1 = T^0 \cap \forall_1 = B \cap \forall_1,$$

and hence

$$B^0 = T^0 \subseteq G.$$

Theorem 3.2 now gives

$$\neg \text{CON(B)} \in G \subseteq N^g$$

which is a contradiction since

$$\text{CON(B)} \in N^0 \subseteq N^g.$$

This completes the proof.

Clearly we have $\mathfrak{N} \in \mathscr{F}$ and so $E \subseteq F \subseteq N$. From theorem 2.1(ii) we see that N is axiomatized by $E \cup \{\rho\}$, and so that

$$\sigma \mapsto \rho \to \sigma$$

is an interpretation of N in E. However, there is a more useful interpretation.

For each formula ϕ let $\phi^{(I)}$ be the formula obtained from ϕ by relativizing all quantifiers to I. Theorem 2.1(i) shows that for each $\mathfrak{A} \in \mathscr{E}$, each formula ϕ, and each ω-sequence $\bar{n}$ of members of ω,

$$\mathfrak{N} \models \phi(\bar{n}) \Leftrightarrow \mathfrak{A} \models \phi^{(I)}(\bar{n}).$$

Thus the map

$$\sigma \mapsto \sigma^{(I)}$$

is an interpretation of N in (all extensions of) E.

A set $X \subseteq \omega$ is definable in $\mathfrak{A} \in \mathscr{E}$ if for some formula ϕ,

$$n \in X \Leftrightarrow \mathfrak{A} \models \phi(n)$$

holds for all $n \in \omega$. Let us say a degree $\boldsymbol{d}$ is represented in $\mathfrak{A}$ if some member of $\boldsymbol{d}$ is definable in $\mathfrak{A}$. The set $d(\mathfrak{A})$ of degrees represented in $\mathfrak{A}$ is of interest.

The equivalence (3.1) gives us the following theorem.

THEOREM 3.4. i) N *is interpretable in* E *and all its extensions.*

ii) *For each number theory* T, *if* E $\subseteq$ T *then* T *is not arithmetical.*

iii) *Each arithmetical* $X \subseteq \omega$ *is definable in each* $\mathfrak{A} \in \mathscr{E}$.

iv) *If* X *is arithmetical in* Y *(both being subsets of* ω*) and* Y *is definable in* $\mathfrak{A} \in \mathscr{E}$, *then* X *is definable in* $\mathfrak{A}$.

v) *For each* $\mathfrak{A} \in \mathscr{E}$, *if* $\boldsymbol{d} \in d(\mathfrak{A})$, *then each member of* $\boldsymbol{d}$ *is definable in* $\mathfrak{A}$.

vi) *For each* $\mathfrak{A} \in \mathscr{E}$, $d(\mathfrak{A})$ *is an initial segment of degrees, contains all arithmetical degrees, and is closed under the jump operation.*

If we replace the pair E, $\mathscr{E}$ by the pair G, $\mathscr{C}$, the results of Theorem 3.4 can be considerably improved. Roughly speaking, we can replace N by full second order number theory 2N. We obtain an interpretation of 2N in G from the following theorem.

THEOREM 3.5. *Let* $\mathfrak{A} \in \mathscr{C}$, $X \subseteq \omega$. *There is some* $a \in A$ *such that*

$$n \in X \Leftrightarrow \mathfrak{A} \models f(n, a)$$

holds for all $n \in \omega$.

PROOF. Consider the countable type (in the sense of §1)

$$\Gamma(w) = \{f_+(n, w) : n \in X\} \cup \{\neg f_-(n, w) : n \notin X\}.$$

This is finitely satisfiable in any number structure; in particular, we have some $\mathfrak{B} \succ \mathfrak{A}$ which satisfied Γ. Thus, since $\mathfrak{A} \in \mathscr{C}$, Γ is satisfied in $\mathfrak{A}$, as required.

Theorem 3.5 shows that the elements of $\mathfrak{A} \in \mathscr{C}$ can be used to index the subsets of ω (in the same way that the elements of $\mathfrak{N}$ can be used to index the finite subsets of ω), so we can replace second order quantification over ω by first order quantification over A. For each second order formula Φ containing no free second order variables, there is a first order formula ϕ containing the same free variables as Φ, such that for each $\mathfrak{A} \in \mathscr{C}$ and each ω-sequence $\bar{n}$ of members of ω,

$$\mathfrak{N} \models \Phi(\bar{n}) \Leftrightarrow \mathfrak{A} \models \phi(\bar{n}). \tag{3.2}$$

The associated map

$$\Sigma \mapsto \sigma$$

from second order sentences to first order sentences is an interpretation of 2N in (all extensions of) G.

For each $\mathfrak{A} \in \mathscr{C}$ we define in the obvious way the set $D(\mathfrak{A})$ of hyperdegrees $\boldsymbol{D}$ represented in $\mathfrak{A}$.

The equivalence (3.2) gives us the following theorem.

THEOREM 3.6. i) 2N *is interpretable in* G *and all its extensions.*

ii) *For each number theory* T, *if* G $\subseteq$ T *then* T *is not analytical.*

iii) *Each analytical* $X \subseteq \omega$ *is definable in each* $\mathscr{C} \in \mathscr{C}$.

iv) *If* X *is analytical in* Y *(both being subsets of* ω*) and* Y *is definable in* $\mathfrak{A} \in \mathscr{C}$, *then* X *is definable in* $\mathfrak{A}$.

v) *For each* $\mathfrak{A} \in \mathscr{C}$, *if* $\boldsymbol{D} \in D(\mathfrak{A})$ *then each member of* $\boldsymbol{D}$ *is definable in* $\mathfrak{A}$.

vi) *For each* $\mathfrak{A} \in \mathscr{C}$, $D(\mathfrak{A})$ *is an initial segment of hyperdegrees, contains all analytical hyperdegrees, and is closed under the hyperjump operation.*

The next theorem gives a kind of normal form for formulae over G.

THEOREM 3.7. *For each formula* $\phi(u_1, \cdots, u_r, v)$ *the sentence*

$$(\forall u_1, \cdots, u_r)(\exists w)(\forall v \in I)[\phi(u_1, \cdots, u_r, v) \leftrightarrow f(v, w)]$$

is in G.

PROOF. Consider any $\mathfrak{A} \in \mathscr{C}$, any elements $a_1, \cdots, a_r$ of $\mathfrak{A}$, and let

$$X = \{n \in \omega : \mathfrak{A} \models \phi(a_1, \cdots, a_r, n)\}.$$

Theorem 3.5 gives us some $a \in A$ such that

$$\mathfrak{A} \models \phi(a_1, \cdots, a_r, n) \leftrightarrow f(n, a)$$

holds for all $n \in \omega$. Thus, since $I^{\mathfrak{A}} = \omega$,

$$\mathfrak{A} \models (\exists w)(\forall v \in I)[\phi(a_1, \cdots, a_r, v) \leftrightarrow f(v, w)],$$

which gives the required result.

There is no similar normal form for F, but we do have the following theorem.

THEOREM 3.8. *For each formula* $\phi(v)$ *and formula* $\varepsilon(v, w) \in \exists_1$, *the sentence*

$$(\exists w)(\forall v \in I)[\phi(v) \leftrightarrow \varepsilon(v, w)]$$

$$\rightarrow \quad (\exists w \in I)(\forall v \in I)[\phi(v) \leftrightarrow d(v, w)]$$

is in F.

PROOF. Consider any $\mathfrak{A} \in \mathscr{F}$ such that

$$\mathfrak{A} \models (\exists w)(\forall v \in I)[\phi(v) \leftrightarrow \varepsilon(v, w)].$$

Thus we have some $a \in A$ with $\mathfrak{A} \models \psi(a)$, where

$$\psi(w) = (\forall v \in I)[\phi(v) \leftrightarrow \varepsilon(v, w)].$$

Remembering that $\mathfrak{A} \in \mathscr{F}$, a simple compactness argument produces a formula $\phi(w) \in \exists_1$ such that $\mathfrak{A} \models \phi(a)$ and

$$\mathfrak{A} \models (\forall w)[\theta(w) \to \psi(w)].$$

We now consider the $\forall_1$-formula

$$\chi(v) = (\exists w)[\theta(w)_\wedge \varepsilon(v, w)],$$

for which

$$\mathfrak{A} \models \phi(n) \leftrightarrow \chi(n)$$

holds for all $n \in \omega$. We may replace $\chi(v)$ by $d(v, m)$ for some $m \in \omega$, and so

$$\mathfrak{A} \models (\exists w \in I)(\forall v \in I)[\phi(v) \leftrightarrow d(v, w)],$$

which gives the required result.

Combining Theorems 3.7 and 3.8, we get the following theorem.

THEOREM 3.9. *There is an* $\forall_4$*-sentence* σ *such that* $\sigma \in$ F *and* $\neg\sigma \in$ G.

PROOF. Let $\delta(v)$ be any $\forall_1$-formula satisfying (2.2), and let σ be

$$\neg(\exists w)(\forall v \in I)[\delta(v) \leftrightarrow f(v, w)].$$

Clearly $\sigma \in \forall_4$, and Theorem 3.7 gives $\neg\sigma \in$ G.

Consider any $\mathfrak{A} \in \mathscr{F}$, and suppose that $\mathfrak{A} \models \neg\sigma$. Theorem 3.8 now gives

$$\mathfrak{A} \models (\exists w \in I)(\forall v \in I)[\delta(v) \leftrightarrow d(v, w)]$$

which contradicts (2.2), and so $\mathfrak{A} \models \sigma$. Thus $\sigma \in$ F, as required.

Finally we prove the following theorem.

THEOREM 3.10. *For each number theory* T, *the only inclusions between the theories* E, F, G, T^e, T^f, T^g *are the obvious ones,*

$$\begin{array}{ccc} F & \subseteq & T^f \\ \cup\!| & & \cup\!| \\ E & \subseteq & T^e \\ |\!\cap & & |\!\cap \\ G & \subseteq & T^g. \end{array}$$

In particular these six theories are distinct.

PROOF. There are $6 \times 5 = 30$ possible inclusions between the six theories Of these inclusions, nine hold (as indicated) and 16 fail because of Theorem 3.9 (F, G have no common extension). The remaining five

$$T^e \subseteq E,\ T^f \subseteq F,\ T^g \subseteq G,$$

$$T^e \subseteq F,\ T^e \subseteq G,$$

fail because of Theorem 3.3.

4. Some open questions

In this last section, we ask several questions which are either interesting in themselves or would increase our understanding of forcing companions. Several of these questions are not particularly concerned with number theories, although they are probably most easily answered for number theories.

1. Which theories are such that all their companions are equal? Clearly T is such if and only if T^f is $\forall_2$-axiomatizable, but this is not a useful characterization.

2. Does Theorem 1.1 have a converse? The full converse is false since there are complete, non-model-complete, $\forall_2$-axiomatizable theories.

3. The space of companion operators is ordered in a natural way, and is a lower semi-lattice. What does this lattice look like? What are its universal and homogeneous properties?

4. A companion operator is, in general, little more than a choice function, but the interesting ones are "uniform" in some sense. Can this be made precise?

5. Suppose $T_1 \subseteq T_2$ and $(-)^*$ is a "natural" companion operator. How are T_1^*, T_2^* related?

6. The Stone space of the Lindenbaum algebra of B is a certain topology on the set of complete extensions of B. The closed sets of this topology correspond to the members of **B**. What are the topological properties of companion operators, in particular $(-)^f$?

7. Under what conditions does

$$T_1 \subseteq T_2,\ T_1\, f\text{-complete} \Rightarrow T_2\, f\text{-complete}$$

hold? We know this holds if T_2 is a finite extension of T_1.

8. Let $\mathscr{K}$ be any of the classes $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{E}_T, \mathscr{F}_T, \mathscr{G}_T$, and let $i(\mathscr{K}), j(\mathscr{K}), k(\mathscr{K})$ be the number of countable isomorphism types represented in $\mathscr{K}$, the number of

elementary types represented in $\mathscr{K}$, the least cardinal K such that $\mathscr{K}$ is $L_{\kappa,\omega}$ definable, respectively. What can $i(\mathscr{K})$, $j(\mathscr{K})$, $k(\mathscr{K})$ be? Apart from the obvious inequalities the following are known or conjectured.

i) T has JEP if and only if $j(\mathscr{F}_T) = 1$ if and only if $j(\mathscr{G}_T) = 1$.

ii) T has a model companion if and only if $k(\mathscr{E}_T) = \aleph_0$ if and only if $k(\mathscr{F}_T) = \aleph_0$ if and only if $k(\mathscr{G}_T) = \aleph_0$.

iii) $k(\mathscr{K}) \leqq (2^{\aleph_0})^+$; $k(\mathscr{E}_T)$, $k(\mathscr{F}_T) \leqq \aleph_1$.

iv) For each number theory T, $i(\mathscr{G}_T) = 2^{\aleph_0}$.

v) For each complete number theory T, $i(\mathscr{F}_T) = 1$. This is a result of Hirschfeld.

vi) For each r.e. number theory T, $i(F_T) = 2^{\aleph_0}$.

vii) For each number theory T, if T does not have JEP then $j(\mathscr{F}_T) = j(\mathscr{G}_T) = 2^{\aleph_0}$.

viii) $j(\mathscr{E}) = j(\mathscr{F}) = j(\mathscr{G}) = 2^{\aleph_0}$.

9. What are the properties of the theories E, F, G and the theory

$$O = \cap \{T^0 : T \in \mathbf{B}\}?$$

We know that N is axiomatized by $E \cup \{\rho\}$ or $F \cup \{\rho\}$. Using Hirschfeld's work we can construct $\tau \in \forall_4$ such that F is axiomatized by $E \cup \{\tau\}$; this sentence also satisfies $\neg \tau \in G$. We believe there is a sentence $\sigma \in F \cap G$ such that $\sigma \notin T^e$ for each number theory T.

Postscript

Using the method of Hirschfeld, [3], we can obtain an $I \in \exists_2$. This gives several improvements to our quantifier bounds, as follows.

Theorem 2.1: $I \in \exists_2$, $\rho \in \forall_3$.

Theorem 2.5: $\rho_0 \in \forall_3$, $\rho_n \in \forall_{n+3}$.

Theorem 3.9: $\sigma \in \forall_3$.

§4, Remark 9: $\tau \in \forall_3$.

In fact this new I is of the form

(bounded $\exists$)($\forall$)(quantifier free)

so there is some $I' \in \forall_1$ with

$$\mathrm{P} \vdash (\forall x)[I(x) \leftrightarrow I'(x)].$$

However, this equivalence is not provable in B, for otherwise we would have some $\rho \in \forall_2$ which is not possible.

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